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Dendritic growth from a melt in an external flow: uniformly valid asymptotic solution for the steady state

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This work is part of an investigation dealing with the effect of external flow in a melt on dendritic growth. In this paper, we will consider steady growth with zero surface tension. Assuming that the Prandtl number Pr is large, we are able to obtain a uniformly valid asymptotic solution for the steady state in the whole growth region.

1. Introduction

The interaction of convection and dendritic growth has been a subject of great interest in the area of materials science in recent years. The experimental observations have shown that convective motion in liquid may have a significant effect on dendritic growth. The existence of the convection may significantly change the growth velocity of the tip and the micro-structure pattern. The preliminary investigations of dendrite growth with convection are usually focused on the special case of zero surface tension. The solution for this special case, as the Ivantsov (1947) solution for dendrite growth with no convection, cannot resolve the problems of the selection of the tip velocity of dendrite and the dynamics of pattern formation on the interface. However, this solution is expected to provide a basis of further investigations on the general case of dendrite growth with non-zero surface tension.

In the past few years, theoretical studies of the steady dendrite growth in an external flow have been conducted by a number of authors, including Ananth & Gill (1989, 1991), Benamar, Bouisson & Pelce (1988), Saville & Beaghton (1988), and also Dash & Gill (1984), Ben Amar *et al.* (1988), Bouissou & Pelce (1989), Saville & Beaghton (1988), McFadden & Coriell (1986), Ananth & Gill (1989, 1991), Canright & Davis (1990) numerically and analytically. These authors considered the special case of zero surface tension, and obtained similarity solutions for the problem by use of some simplifying models: Stokes flow model, Oseen flow model, etc. When the Navier–Stokes equations are adopted, the similarity solutions may still be considered as the approximate solutions in the limiting case $Pr \to \infty$. However, as we know, only in the near field can the Stokes model be considered a good approximation to the Navier–Stokes model, whereas only in the far field is the Oseen model a good approximation to the Navier–Stokes model. Therefore, their similarity solutions cannot be considered good approximations in the whole physical space, as far as the Navier–Stokes model is concerned. In other words, for the problem with Navier–Stokes equations, these authors' analytical solutions cannot be used as the first part of a uniformly valid asymptotic expansion solution. Moreover, the approaches adopted by these authors do not allow the generation of the next-order approximations, nor give an estimation of the error between their solutions and the exact solutions.

Our project attempts to investigate the effect of convection motion induced by external flow on dendrite growth, based on the Navier–Stokes model. We wish to find uniformly valid asymptotic solutions for the problem. In the present paper, we restrict ourselves to the case of steady dendrite growth with zero surface tension. On the basis of this steady-state solution, we shall, in a future paper, study the unsteady perturbed state with isotropic surface tension, explore the effect of convective motion on the global instability mechanism and resolve the problems of the selection of tip velocity and pattern formation in solidification.

Evidently, with the inclusion of convective motion induced by the external flow, the system becomes more complicated. Our study shows that even for the special case of zero surface tension, the system no longer allows an exact similarity solution. (This conclusion has been also drawn by Ananth & Gill 1991 in terms of a different approach.) However, when the Prandtl number Pr is large, the system does allow a nearly similar solution with the error of $O(1/(Pr \ln Pr))$. This nearly similar solution differs from the classic Ivantsov solution, with the correction terms proportional to the parameter δ_0. The parameter δ_0 is a function of the Prandtl number Pr, flow parameter U_∞ and undercooling T_∞. It tends to zero as $Pr \to \infty$ or $U_\infty \to 0$.

The present paper is arranged as follows. In §2, we present the mathematical formulation of the problem; in §3, we obtain the zeroth-order inner asymptotic expansion solution as $Pr \to \infty$; in §4, we derive the outer asymptotic expansion solutions in the outer region; in §5, we derive the higher-order asymptotic solutions, and match the inner solutions with the outer solutions; in §6, we derive the asymptotic expansion solutions for the temperature field and the interface shape; finally, in §7, we summarize the results and draw some conclusions.

2. Mathematical formulation of the problem

Consider a single dendrite growing into an undercooled pure melt in the negative z-direction with a constant average velocity U as shown in figure 1. We assume that in the far field ahead of dendrite, the melt flows along the z-direction with a constant velocity $(U_\infty)_D$. Both the growth velocity U and the flow velocity $(U_\infty)_D$ are measured in the laboratory frame. For simplicity, we assume that the density ρ and the other thermal characteristic constants of the solid phase, such as thermal diffusivity κ_T and the specific heat c_p, are the same as the corresponding quantities of the liquid phase. Gravity is taken to be negligible, the surface tension is assumed isotropic, so then the dendrite is axisymmetrical; and no convective motion in the system is induced by any source other than the external flow. We utilize the thermal length $l_T = \kappa_T/U$ as the lengthscale, l_T/U as the timescale and $\Delta H/(c_p \rho)$ as the temperature scale, where ΔH is the latent heat per unit volume of the solid. Evidently, the convective motion in the melt will affect the heat transport process and change the temperature distribution.

We use a moving paraboloidal coordinate system (ξ, η, θ) fixed at the dendrite tip, which is defined, in terms of the cylindrical coordinate system (r, z, θ), as follows (see figure 2):

$$r/\eta_0^2 = \xi\eta, \qquad z/\eta_0^2 = \tfrac{1}{2}(\xi^2 - \eta^2), \tag{2.1}$$

and θ is the azimuthal angle. In (2.1), the parameter η_0^2 is introduced to normalize the interface shape function. For any given undercooling T_∞, we can properly choose η_0^2, such that the basic Ivantsov solution has the interface shape $\eta_* = 1$. It is evident that the parameter η_0^2 is just the Péclet number in Ivantsov's solution. Let $\{V(\xi, \eta, t)$ and $V_S(\xi, \eta, t)\}$ represent the absolute velocity field in the liquid state and the solid state,

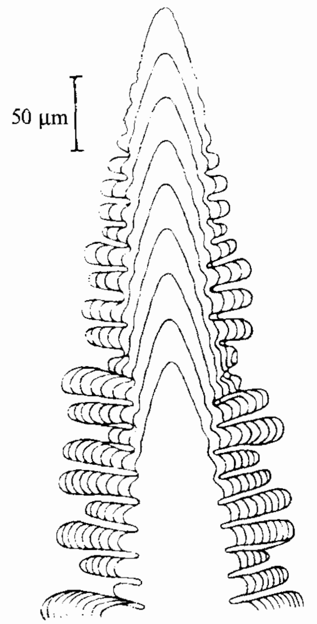

FIGURE 1. A typical dendrite growing from a supersaturated solution.

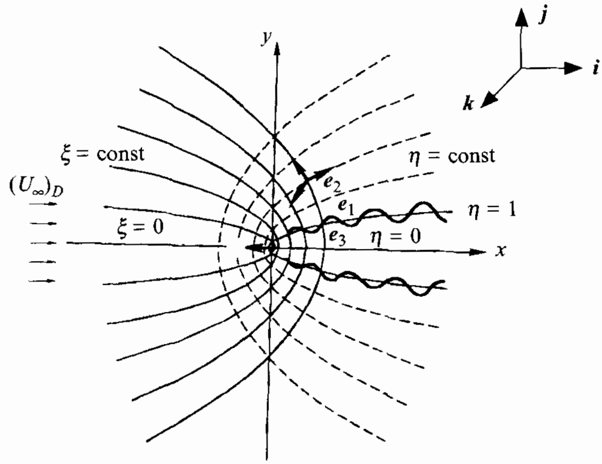

FIGURE 2. The paraboloidal coordinate system used in the present paper.

respectively. Let $V = (u, v, w)$, and (u, w) denote the component of the absolute velocity along the ξ- and η-directions, respectively, in the moving frame at the instant t. Furthermore, let $\eta_s(\xi, t)$ denote the interface shape function; $\Omega = (0, \omega_2, 0) = \nabla \times V$ denote the vorticity; $\Psi(\xi, \eta, t)$ denote the stream function; and T and T_S denote the temperature field in the melt and in the solid state, respectively. The subscript S refers to the solid state. The melt is considered an incompressible Newtonian fluid. The governing equations for the dendritic growth process consist of the fluid dynamical equations and the heat conduction equation. We use the stream function $\Psi(\xi, \eta, t)$ and the vorticity $\Omega = (0, \omega_2 = \zeta/\eta_0^2\,\xi\eta, 0)$ as the basic hydrodynamical quantities. Thus,

$$u = \frac{1}{\eta_0^4\,\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial \Psi}{\partial \eta}, \qquad w = -\frac{1}{\eta_0^4\,\xi\eta(\xi^2+\eta^2)^{\frac{1}{2}}}\frac{\partial \Psi}{\partial \xi}, \tag{2.2}$$

and the non-dimensional governing equations can be written in the following forms:

kinematic equation

$$D^2\Psi = \eta_0^4(\xi^2+\eta^2)\,\zeta; \tag{2.3}$$

vorticity equation

$$Pr\,D^2\zeta = \eta_0^4(\xi^2+\eta^2)\frac{\partial\zeta}{\partial t}+\frac{2\zeta}{\eta_0^3\,\xi^2\eta^2}\frac{\partial(\Psi,\eta_0^2\,\xi\eta)}{\partial(\xi,\eta)}-\frac{1}{\eta_0\,\xi\eta}\frac{\partial(\Psi,\zeta)}{\partial(\xi,\eta)}; \tag{2.4}$$

heat conduction equation

$$\nabla^2 T = \eta_0^4(\xi^2+\eta^2)\frac{\partial T}{\partial t}+\frac{1}{\eta_0^2\,\xi\eta}\left(\frac{\partial\Psi}{\partial\eta}\frac{\partial T}{\partial\xi}-\frac{\partial\Psi}{\partial\xi}\frac{\partial T}{\partial\eta}\right). \tag{2.5}$$

Here, the differentiation operators ∇^2 and D^2 are defined as

$$\nabla^2 = \left\{\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{1}{\eta}\frac{\partial}{\partial\eta}\right\}, \tag{2.6}$$

$$D^2 = \left\{\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-\frac{1}{\xi}\frac{\partial}{\partial\xi}-\frac{1}{\eta}\frac{\partial}{\partial\eta}\right\}. \tag{2.7}$$

The boundary conditions are:

as $\eta\to\infty$,

$$\Psi\to\tfrac{1}{2}\eta_0^4(1+U_\infty)\,\xi^2\eta^2;\quad \zeta\to 0;\quad (\text{or } u\to 0;\quad w\to -1), \tag{2.8}$$

$$T\to T_\infty; \tag{2.9}$$

as $\eta\to 0$,

$$\frac{\partial T_S}{\partial\xi} = 0;\quad T_S = O(1); \tag{2.10}$$

the interface condition at $\eta = \eta_s(\xi, t)$, given as follows:

thermo-dynamical equilibrium condition

$$T = T_S; \tag{2.11}$$

Gibbs–Thomson condition

$$T_S = -\frac{\Gamma}{\eta_0^2}K\left\{\frac{d^2}{d\xi^2},\frac{d}{d\xi}\right\}\eta_s(\xi,t); \tag{2.12}$$

where the curvature operator

$$K\left\{\frac{d^2}{d\xi^2},\frac{d}{d\xi}\right\}\eta_s = -\frac{1}{(\xi^2+\eta_s^2)^{\frac{1}{2}}}\left\{\frac{\eta_s''}{(1+\eta_s'^2)^{\frac{3}{2}}}-\frac{1}{\eta_s(1+\eta_s'^2)^{\frac{1}{2}}}+\frac{\eta_s'(\eta_s^2+2\xi^2)-\xi\eta_s}{\xi(\xi^2+\eta_s^2)(1+\eta_s'^2)^{\frac{1}{2}}}\right\}; \tag{2.13}$$

enthalpy conservation condition

$$\left(\frac{\partial T}{\partial\eta}-\eta_s'\frac{\partial T}{\partial\xi}\right)-\left(\frac{\partial T_S}{\partial\eta}-\eta_s'\frac{\partial T_S}{\partial\xi}\right)+\eta_0^2(\xi\eta_s)'+\eta_0^4(\xi^2+\eta_s^2)\frac{\partial\eta_s}{\partial t} = 0; \tag{2.14}$$

mass conservation condition

$$\left(\frac{\partial\Psi}{\partial\xi}+\eta_s'\frac{\partial\Psi}{\partial\eta}\right) = \eta_0^4(\xi\eta_s)(\xi\eta_s)'; \tag{2.15}$$

continuity condition of the tangential component of velocity

$$\left(\frac{\partial \Psi}{\partial \eta}-\eta_s'\frac{\partial \Psi}{\partial \xi}\right)+\eta_0^4(\xi\eta_s)(\eta_s\eta_s'-\xi)=0. \tag{2.16}$$

In the above, the prime represents the derivative with respect to ξ; the surface tension parameter Γ is defined as

$$\Gamma = l_c/l_T, \tag{2.17}$$

$$l_c=\frac{\gamma c_p T_{M0}}{(\Delta H)^2}, \tag{2.18}$$

where γ is the surface tension, T_{M0} is the melting temperature at the flat interface and l_c is the capillary length; the undercooling parameter T_∞ is defined as

$$T_\infty=\frac{(T_\infty)_D}{\Delta H/(c_p\rho)}, \tag{2.19}$$

while the parameter

$$U_\infty=\frac{(U_\infty)_D}{U}, \tag{2.20}$$

which measures the strength of the external flow. The Prandtl number, Pr, is defined as

$$Pr=\nu/\kappa_T, \tag{2.21}$$

which is equivalent to the inverse of the Reynolds number $Re = Ul_T/\nu$ based on the tip velocity U and the thermal length l_T.

3. Zeroth-order inner solution for the steady state

In this paper, we shall study the steady state of dendrite growth with zero surface tension. Thus, in (2.2)–(2.16), the surface tension parameter $\Gamma = 0$, and $\partial/\partial t = 0$.

It is noticed that the following Ivantsov similarity solution with no external flow is a particular solution of (2.2)–(2.5):

$$\begin{gathered} T_* = T_\infty + b\,\mathrm{e}^b E_1(b\eta^2), \quad T_{S*}=0, \\ \eta_*=1, \quad \zeta_*=0, \quad \Psi_*=\tfrac{1}{2}\eta_0^4\xi^2\eta^2, \end{gathered} \tag{3.1}$$

where $E_1(x)$ is the exponential function (see Abramovitz & Stegun 1964), and

$$-T_\infty = b\,\mathrm{e}^b E_1(b^2), \quad b=\tfrac{1}{2}\eta_0^2. \tag{3.2}$$

It is seen that the parameter η_0^2 is now determined by the undercooling T_∞. We assume that the Prandtl number of the system, Pr, is very large. Thus, for any fixed (ξ,η) as $Pr\to\infty$ we can make the following asymptotic expansions:

$$\left.\begin{aligned} \Psi(\xi,\eta) &= \Psi_* + \Delta_0(Pr)\,\Psi_0(\xi,\eta)+\Delta_1(Pr)\,\Psi_1(\xi,\eta)+\dots, \\ \zeta(\xi,\eta) &= \zeta_* + \Delta_0(Pr)\,\zeta_0(\xi,\eta)+\Delta_1(Pr)\,\zeta_1(\xi,\eta)+\dots, \\ T(\xi,\eta) &= T_* + \Delta_0(Pr)\,T_0(\xi,\eta)+\Delta_1(Pr)\,T_1(\xi,\eta)+\dots, \\ T_S(\xi,\eta) &= T_{S*}\Delta_0(Pr)\,T_{S0}(\xi,\eta)+\Delta_1(Pr)\,T_{S1}(\xi,\eta)+\dots, \\ \eta_s &= 1+\Delta_0(Pr)\,h_0+\Delta_1(Pr)\,h_1+\dots, \end{aligned}\right\} \tag{3.3}$$

where $\Delta_0(Pr), \Delta_1(Pr), \dots$ are asymptotic sequences to be determined. However, it will be seen that the above types of expansion cannot satisfy the fluid dynamic conditions in the far field, although they can satisfy all conditions at the interface. Thus, in order to solve the problem under investigation, it is necessary for us to use the matched asymptotic expansion method, by which the whole physical space is divided into the inner region near the interface and the outer region in the far field. The solution has different asymptotic expansion forms in the different regions. The inner and outer asymptotic expansion solutions must be solved first, and then be matched in the intermediate region.

In the following sections, we shall follow this procedure. The expansion (3.3) is just the inner expansion form. In order to obtain the inner solution, we substitute the above expansions into (2.1)–(2.5). Then, we can derive each order of inner expansion solutions successively. In this section, we attempt to derive the zeroth order inner expansion solution $\Delta_0(Pr)\,\Psi_0(\xi,\eta)$.

The zeroth-order inner solution is subject to the following equations:

$$D^2\Psi_0 = -\eta_0^4(\xi^2+\eta^2)\,\zeta_0, \tag{3.4}$$

$$D^2\zeta_0 = 0, \tag{3.5}$$

$$\nabla^2 T_0 = \frac{1}{\eta_0^2\,\xi\eta}\left[\frac{\partial\Psi_*}{\partial\eta}\frac{\partial T_0}{\partial\xi}-\frac{\partial\Psi_*}{\partial\xi}\frac{\partial T_0}{\partial\eta}+\frac{\partial\Psi_0}{\partial\eta}\frac{\partial T_*}{\partial\xi}-\frac{\partial\Psi_0}{\partial\xi}\frac{\partial T_*}{\partial\eta}\right], \tag{3.6}$$

with the boundary conditions:

$$\text{as } \eta\to\infty,\quad T_0,\zeta_0\to 0; \tag{3.7}$$

$$\text{as } \eta\to 0,\quad T_S = O(1); \tag{3.8}$$

on the interface $\eta = 1$:

$$T_0 = -T_*'(1)\,h_0 = \eta_0^2\,h_0, \tag{3.9}$$

$$\frac{\partial T_0}{\partial\eta}(\xi,1) = -T_*''(1)\,h_0-\eta_0^2 h_0-\eta_0^2\,\xi h_0'(\xi) = -\eta_0^2(2+\eta_0^2)\,h_0-\eta_0^2\,\xi h_0'(\xi), \tag{3.10}$$

$$\Psi_0(\xi,1) = 0, \tag{3.11}$$

$$\frac{\partial}{\partial\eta}\Psi_0(\xi,1) = 0. \tag{3.12}$$

To solve the above system (3.4)–(3.12), one must first obtain the velocity field from (3.4) and (3.5) with the boundary conditions (3.11) and (3.12); then obtain the temperature field and the interface shape from (3.6) and the boundary conditions (3.7)–(3.10). To solve (Ψ_0,ζ_0), it is convenient to utilize a set of new variables defined as follows:

$$\sigma = \tfrac{1}{2}\eta_0^2\,\xi^2 \quad \tau = \tfrac{1}{2}\eta_0^2\,\eta^2. \tag{3.13}$$

By using the variables $(\sigma;\tau)$, the operator D^2 becomes

$$D^2 = 2\eta_0^2\,L, \tag{3.14}$$

where

$$L = \left(\sigma\frac{\partial^2}{\partial\sigma^2}+\tau\frac{\partial^2}{\partial\tau^2}\right). \tag{3.15}$$

Thus, we obtain

$$L[\psi_0] = -(\sigma+\tau)\,\zeta_0, \tag{3.16}$$

$$L[\zeta_0] = 0. \tag{3.17}$$

In general, (3.16) and (3.17) allow the following form of the solutions ζ_0 and Ψ_0:

$$\zeta_0 = \sigma^n f_n(\tau)+\sigma^{(n-1)}f_{n-1}(\tau)+\ldots+f_0(\tau), \tag{3.18}$$

$$\Psi_0 = g_0(\tau)+\sigma g_1(\tau)+\ldots+\sigma^n g_n(\tau), \tag{3.19}$$

where $f_i(\tau)$, $g_i(\tau)$ $(i = 0, 1, 2, \ldots, n)$ are polynomials of τ. However, it is verified that for the zeroth-order inner solution, one only need to take $n = 1$. By substituting (3.18) and (3.19) into (3.16) and (3.17), we obtain the solution

$$\left.\begin{aligned} \zeta_0 &= a_0, \\ \Psi_0 &= -\left(\frac{a_0\tau^2}{2} + d_1\tau + d_2\right) + \sigma[-a_0\tau\ln\tau + (a_0+d_3)\tau + d_4]. \end{aligned}\right\} \tag{3.20}$$

The boundary conditions (3.11) and (3.12) determine the arbitrary constants $\{d_1, d_2, d_3, d_4\}$ in (3.20) in terms of a_0. Consequently, we derive the inner solution

$$\Psi_0(\xi, \eta) = a_0\,\psi(\sigma, \tau), \tag{3.21}$$

where

$$\begin{aligned} \psi &= -(\tfrac{1}{2}\tau^2 - \tfrac{1}{2}\eta_0^2\tau + \tfrac{1}{8}\eta_0^4) + \sigma[-\tau\ln\tau + \tau(1+\ln\tfrac{1}{2}\eta_0^2) - \tfrac{1}{2}\eta_0^2] \\ &= -\tfrac{1}{4}\eta_0^4\{\tfrac{1}{2}\eta^4 - \eta^2 + \tfrac{1}{2} - \xi^2[-\eta^2\ln\eta^2 + \eta^2 - 1]\}. \end{aligned} \tag{3.22}$$

The above solution contains an arbitrary constant a_0. As we pointed out before, this solution satisfies all the boundary conditions at the interface. It, however, cannot satisfy the far-field condition as $\eta \to \infty$. It is for this reason that we call this solution the inner solution. In the far field, the inner expansion solution is not valid; the steady-state solution has a different asymptotic expansion, which we call the outer expansion solution. The inner expansion solution and the outer expansion solution, of course, must match with each other in the intermediate region. The gauge function $\Delta_0(Pr)$ and the arbitrary constant a_0 in the inner solution Ψ_0 will be determined by the matching condition (see Kevorkian & Cole 1981). Therefore, it will be seen later that the zeroth-order inner solution will be affected by the far-field condition through the constant a_0.

4. Outer expansion solution

Now, we turn to the study of the outer asymptotic expansion solutions in the far field for the system (2.2)–(2.16). We introduce the following outer variables (ξ_*, η_*):

$$\xi_* = \xi/Pr^{\frac{1}{2}} \quad \eta_* = \eta/Pr^{\frac{1}{2}}, \tag{4.1}$$

and make the following outer asymptotic expansion for fixed (ξ_*, η_*) and in the limit as $Pr \to \infty$:

$$\left.\begin{aligned} \Psi(\xi,\eta) &= \Psi_* + \mu_0(Pr)\,\hat{\Psi}_0(\xi,\eta) + \mu_1(Pr)\,\hat{\Psi}_1(\xi,\eta) + \ldots, \\ \zeta(\xi,\eta) &= \frac{1}{Pr^2}\{\zeta_* + \mu_0(Pr)\,\hat{\zeta}_0(\xi,\eta) + \mu_1(Pr)\,\hat{\zeta}_1(\xi,\eta) + \ldots\}, \\ T(\xi,\eta) &= T_* + \mu_0(Pr)\,\hat{T}_0(\xi,\eta) + \mu_1(Pr)\,\hat{T}_1(\xi,\eta) + \ldots. \end{aligned}\right\} \tag{4.2}$$

In terms of the outer variables, (2.3)–(2.4) are transformed to the form

$$\mathrm{D}_*^2\,\hat{\Psi} = Pr^2\eta_0^4(\xi_*^2 + \eta_*^2)\,\hat{\zeta}, \tag{4.3}$$

$$\begin{aligned} \mathrm{D}_*^2\,\hat{\zeta} = {} & \frac{2\hat{\zeta}}{Pr^2\eta_0^2\xi_*^2\eta_*^2}\left(\xi_*\frac{\partial\hat{\Psi}}{\partial\xi_*} - \eta_*\frac{\partial\hat{\Psi}}{\partial\eta_*}\right) \\ & - \frac{1}{Pr^2\eta_0^2\xi_*\eta_*}\left(\frac{\partial\hat{\Psi}}{\partial\xi_*}\frac{\partial\hat{\zeta}}{\partial\eta_*} - \frac{\partial\hat{\Psi}}{\partial\eta_*}\frac{\partial\hat{\zeta}}{\partial\xi_*}\right) - \frac{1}{\eta_0^2\xi_*\eta_*}\left(\frac{\partial\hat{\Psi}}{\partial\xi_*}\frac{\partial\hat{\zeta}}{\partial\eta_*} - \frac{\partial\hat{\Psi}}{\partial\eta_*}\frac{\partial\hat{\zeta}}{\partial\xi_*}\right). \end{aligned} \tag{4.4}$$

The boundary conditions as $\eta_* \to \infty$ are

$$\hat{\Psi}(\xi_*, \eta_*) \approx Pr^2\tfrac{1}{2}\eta_0^4 U_\infty\,\xi_*^2\,\eta_*^2 \tag{4.5}$$

or
$$\frac{1}{\eta_0^4 \xi_* \eta_*^2} \frac{\partial \hat{\Psi}}{\partial \xi_*} \approx Pr^2 U_\infty; \quad \frac{1}{\eta_0^4 \xi_* \eta_*^2} \frac{\partial \hat{\Psi}}{\partial \eta_*} \approx 0. \tag{4.6}$$

To derive the outer expansion solutions, it is more convenient to use the following set of variables:

$$\sigma_* = \tfrac{1}{2}\eta_0^2 \xi_*^2, \quad \tau_* = \tfrac{1}{2}\eta_0^2 \eta_*^2. \tag{4.7}$$

In what follows, we shall derive the outer expansion solutions of the first few orders, successively.

$O(\mu_0(Pr))$: zeroth-order outer solution $\mu_0(Pr)\,\hat{\Psi}_0(\xi_*, \eta_*)$

To satisfy the far-field condition (4.5) as $\eta_* \to \infty$, we must set

$$\mu_0(Pr) = Pr^2 \tag{4.8}$$

and have
$$\hat{\Psi}_0(\xi_*, \eta_*) \approx \tfrac{1}{2}\eta_0^4 U_\infty \xi_*^2 \eta_*^2 \quad \text{as} \quad \eta_* \to \infty. \tag{4.9}$$

It is very easy to verify that the zeroth-order outer expansion solution is simply

$$\hat{\Psi}_0(\xi_*, \eta_*) = \tfrac{1}{2}\eta_0^4 U_\infty \xi_*^2 \eta_*^2 = 2U_\infty \sigma_* \tau_*, \tag{4.10}$$

$$\hat{\zeta}_0(\xi_*, \eta_*) = 0, \tag{4.11}$$

which represents a uniform flow with velocity U_∞.

The first matching problem is how to determine the zeroth inner solution Ψ_0, which can match with the outer uniform flow solution $\hat{\Psi}_0$ in the intermediate region. For this purpose, we rewrite the inner solution Ψ_0 in terms the outer variables σ_*, τ_*. Thus, we have

$$\begin{aligned} \Delta_0(Pr)\,\Psi_0(\xi, \eta) = & -a_0[\Delta_0(Pr)\,Pr^2 \ln Pr]\,\sigma_* \tau_* \\ & + a_0[\Delta_0(Pr)\,Pr^2]\{-\sigma_* \tau_* \ln \tau_* - \tfrac{1}{2}\tau_*^2 + [1 + \ln \tfrac{1}{2}\eta_0^2]\,\sigma_* \tau_*\} \\ & - a_0 \tfrac{1}{2}\eta_0^2[\Delta_0(Pr)\,Pr]\,(\sigma_* - \tau_*) - a_0 \tfrac{1}{8}\eta_0^2[\Delta_0(Pr)]. \end{aligned} \tag{4.12}$$

It is seen that the zeroth-order inner solution $\Delta_0(Pr)\,\Psi_0$ consists of four parts in the right-hand side of (4.12). In order for the first part of the inner solution $\Delta_0(Pr)\,\Psi_0$ to match with the outer solution $Pr^2\hat{\Psi}_0$, we must set

$$\Delta_0(Pr) = 1/\ln Pr, \quad a_0 = -2U_\infty. \tag{4.13}$$

Furthermore, the other three parts of the inner solution $\Delta_0(Pr)\,\Psi_0$ must match with the higher-order outer solutions:

$$\{\mu_1(Pr)\,\hat{\Psi}_1(\sigma_*, \tau_*) + \mu_2(Pr)\,\hat{\Psi}_2(\sigma_*, \tau_*) + \mu_3(Pr)\,\hat{\Psi}_3(\sigma_*, \tau_*) + \ldots\}. \tag{4.14}$$

Specifically, in order for the second part of the inner solution $\Delta_0(Pr)\,\Psi_0$ to match with the outer solution $\mu_1(Pr)\,\hat{\Psi}_1$, we must set

$$\Delta_0(Pr)\,Pr^2 = \mu_1(Pr) \quad \text{or} \quad \mu_1(Pr) = Pr^2/\ln Pr, \tag{4.15}$$

whereas in order for the third and fourth part of the inner solution $\Delta_0(Pr)\,\Psi_0$ to match with the outer solution $\mu_2(Pr)\,\hat{\Psi}_2$ and $\mu_3(Pr)\,\hat{\Psi}_3$ respectively, we must set

$$\mu_2(Pr) = Pr/\ln Pr, \tag{4.16}$$

$$\mu_3(Pr) = 1/\ln Pr \tag{4.17}$$

and derive
$$\hat{\Psi}_2 = \tfrac{1}{2}U_\infty \eta_0^2(\sigma_* + \tau_*), \tag{4.18}$$

$$\hat{\Psi}_3 = \tfrac{3}{4}U_\infty \eta_0^2. \tag{4.19}$$

In the following, we shall derive the outer solutions $\hat{\Psi}_1$.

$O(\mu_1(Pr))$: the first-order outer solution $\hat{\Psi}_1$

The first-order solution is subject to the following equations:

$$\mathrm{D}_*^2 \hat{\Psi}_1 = -\eta_0^4(\xi_*^2+\eta_*^2)\hat{\zeta}_1, \tag{4.20}$$

$$\mathrm{D}_*^2 \hat{\zeta}_1 = -(1+U_\infty)\eta_0^2\left(\eta_*\frac{\partial\hat{\zeta}_1}{\partial\eta_*}-\xi_*\frac{\partial\hat{\zeta}_1}{\partial\xi_*}\right), \tag{4.21}$$

where the differentiation operators D_*^2 are defined as

$$\mathrm{D}_*^2 = \left\{\frac{\partial^2}{\partial\xi_*^2}+\frac{\partial^2}{\partial\eta_*^2}-\frac{1}{\xi_*}\frac{\partial}{\partial\xi_*}-\frac{1}{\eta_*}\frac{\partial}{\partial\eta_*}\right\}. \tag{4.22}$$

By using the variables $(\sigma_*;\tau_*)$, the operator D_*^2 becomes

$$\mathrm{D}_*^2 = 2\eta_0^2 \mathrm{L}_* \tag{4.23}$$

where

$$\mathrm{L}_* = \left(\sigma\frac{\partial^2}{\partial\sigma_*^2}+\tau_*\frac{\partial^2}{\partial\tau_*^2}\right). \tag{4.24}$$

Thus, we derive

$$\mathrm{L}_*[\hat{\Psi}_1] = -(\sigma_*+\tau_*)\hat{\zeta}_1, \tag{4.25}$$

$$\mathrm{L}_*[\hat{\zeta}_1] = -(1+U_\infty)\left(\tau_*\frac{\partial\hat{\zeta}_1}{\partial\tau_*}-\sigma_*\frac{\partial\hat{\zeta}_1}{\partial\sigma_*}\right). \tag{4.26}$$

In general, the above system of equations allows the following type of solution for $\hat{\zeta}_1$ and $\hat{\Psi}_1$:

$$\hat{\zeta}_1 = \sigma_*^n F_n(\tau_*)+\sigma_*^{(n-1)}F_{n-1}(\tau_*)+\ldots+F_0(\tau_*), \tag{4.27}$$

$$\hat{\Psi}_1 = \sigma_*^n G_n(\tau_*)+\sigma_* G_{n-1}(\tau_*)+\ldots+G_0(\tau_*), \tag{4.28}$$

where $F_i(\tau_*), G_i(\tau_*)(i=0,1,2,\ldots,n)$ are some special functions of τ_*. However, it will be seen that to match with the zeroth-order inner solution, one only need to take $n=0$. By substituting (4.27) and (4.28) into (4.25) and (4.26), we obtain the solutions

$$\hat{\zeta}_1 = F_0(\tau_*) = C_1\,\mathrm{e}^{-(1+U_\infty)\tau_*} \tag{4.29}$$

and

$$\hat{\Psi}_1 = G_0(\tau_*)+\sigma_* G_1(\tau_*)$$
$$= A_0+A_1\tau_*+A_2\sigma_*-\frac{C_1}{(1+U_\infty)^2}[e^{-(1+U_\infty)\tau_*}+(1+U_\infty)\sigma_* E_2((1+U_\infty)\tau_*)], \tag{4.30}$$

where (A_0, A_1, A_2, C_1) are arbitrary constants. Obviously, (4.30) satisfy the far-field condition:

$$\frac{1}{\eta_0^4\xi_*\eta_*^2}\frac{\partial\hat{\Psi}_1}{\partial\xi_*}\to 0;\quad \frac{1}{\eta_0^4\xi_*\eta_*^2}\frac{\partial\hat{\Psi}_1}{\partial\eta_*}\to 0. \tag{4.31}$$

In order for the outer solution to match with the inner solution, we need to expand the outer solution in the limit: $\tau_*\to 0$. In terms of the following formula

$$\left.\begin{aligned} \mathrm{e}^{-x} &= 1-x+\tfrac{1}{2}x^2\ldots,\\ E_2(x) &= 1+(\gamma_0-1)x+x\ln x+\ldots,\end{aligned}\right\}\quad \text{as}\quad x\to 0, \tag{4.32}$$

where the Euler constant

$$\gamma_0 = 0.57721\ldots, \tag{4.33}$$

we derive

$$\left(\frac{Pr^2}{\ln Pr}\right)\hat{\Psi}_1(\sigma_*,\tau_*)=\left(\frac{Pr^2}{\ln Pr}\right)\Bigg\{-C_1[\tfrac{1}{2}\tau_*^2+\sigma_*\tau_*\ln\tau_*-\sigma_*\tau_*(1+\ln\tfrac{1}{2}\eta_0^2)]$$

$$+\left[A_0+A_1\tau_*+A_2\sigma_*+A_2\sigma_*-\frac{C_1}{1+U_\infty}(\sigma_*-\tau_*)\right]\Bigg\}$$

$$+\left(\frac{Pr^2}{\ln Pr}\right)\{O(\tau_*^3,\tau_*^4,\ldots,\sigma_*\tau_*^2,\sigma_*\tau_*^3,\ldots,)\}$$

$$-C_1\left(\frac{Pr^2}{\ln Pr}\right)\sigma_*\tau_*(\gamma_0-1+\ln\tfrac{1}{2}[\eta_0^2(1+U_\infty)]). \tag{4.34}$$

It is seen that the outer solution $\hat{\Psi}_1(\sigma_*,\tau_*)$ consists of three parts. In order for the first part to match with the second part of the inner solution $\Delta_0(Pr)\,\Psi_0(\sigma,\tau)$, we must set

$$C_1=-2U_\infty,\quad A_0=-\frac{2U_\infty}{(1+U_\infty)^2},\quad A_1=-A_2=-\frac{2U_\infty}{1+U_\infty}. \tag{4.35}$$

Therefore, we derive the outer solution:

$$\mu_1(Pr)\,\hat{\Psi}_1(\sigma_*,\tau_*)=\left(\frac{Pr^2}{\ln Pr}\right)\frac{2U_\infty}{1+U_\infty}\phi(\sigma_*,\tau_*), \tag{4.36}$$

where

$$\phi(\sigma_*,\tau_*)=\frac{e^{-(1+U_\infty)\tau_*}-1}{1+U_\infty}+\tau_*+\sigma_*E_2((1+U_\infty)\tau_*)-\sigma_*. \tag{4.37}$$

There are still two parts in the outer solution $\mu_1\hat{\Psi}_1$, which remain unmatched. In order to match this remainder we need to find the higher-order inner solutions. This will be done in the next section.

5. Higher-order inner solutions and matching

In order to match the second part of the outer solution $\mu_1\hat{\Psi}_1$, we need to find higher-order inner solutions:

$$\Delta_1(Pr)\,\Psi_1(\sigma,\tau)+\Delta_2(Pr)\,\Psi_2(\sigma,\tau)+\ldots, \tag{5.1}$$

which satisfy the following far-field conditions, respectively:

$$\Delta_1(Pr)\,\Psi_1(\sigma,\tau)\approx\frac{1}{Pr\ln Pr}O(\sigma\tau^2,\tau^3), \tag{5.2}$$

$$\Delta_2(Pr)\,\Psi_2(\sigma,\tau)\approx\frac{1}{Pr^2\ln Pr}O(\sigma\tau^3,\tau^4), \tag{5.3}$$

$$\vdots$$

Therefore, we must set

$$\Delta_1(Pr)=\frac{1}{Pr\ln Pr},\quad \Delta_2(Pr)=\frac{1}{Pr^2\ln Pr},\quad \ldots. \tag{5.4}$$

In the present paper, we shall not derive these higher-order inner solutions.

Finally, in order to match the third part of the outer solution $\mu_1 \hat{\Psi}_1$, we need to find a higher-order inner solution $\Delta_0^{(1)}(Pr)\,\Psi_0^{(1)}(\sigma,\tau)$, which satisfies the far-field condition:

$$\left.\begin{aligned}\Delta_0^{(1)}(Pr)\,\Psi_0^{(1)}(\sigma,\tau) &\approx \left(\frac{Pr^2}{\ln Pr}\right)2U_\infty A\sigma_*\tau_* \\ &= \left(\frac{1}{\ln Pr}\right)2U_\infty A\sigma\tau,\end{aligned}\right\} \tag{5.5}$$

where

$$A = (\gamma_0 - 1 + \ln\tfrac{1}{2}[\eta_0^2(1+U_\infty)]). \tag{5.6}$$

This problem turns out to be the same as the one we solved before for the inner solution $\Delta_0(Pr)\,\Psi_0(\sigma,\tau)$. Hence, we can write

$$\Delta_0^{(1)}(Pr) = \frac{1}{(\ln Pr)^2}, \quad \Psi_0^{(1)}(\xi,\eta) = -2U_\infty A\psi(\sigma,\tau). \tag{5.7}$$

Again, the inner solution $\Delta_0^{(1)}(Pr)\,\Psi_0^{(1)}(\sigma,\tau)$ will induce another outer solution $\mu_1^{(1)}(Pr)\,\hat{\Psi}_1^{(1)}(\sigma_*,\tau_*)$ as the inner solution $\Delta_0(Pr)\,\Psi_0(\sigma,\tau)$ induced the outer solution $\mu_1(Pr)\,\hat{\Psi}_1(\sigma_*,\tau_*)$. Hence, we can also write

$$\mu_1^{(1)}(Pr) = \frac{Pr^2}{(\ln Pr)^2}, \quad \hat{\Psi}_1^{(1)}(\xi_*,\eta_*) = \frac{2U_\infty}{1+U_\infty}A\phi(\sigma_*,\tau_*). \tag{5.8}$$

This matching procedure will be repeated infinitely. In this manner, we eventually derive the following inner expansion solution:

$$\begin{aligned}\Psi(\xi,\eta) &= \tfrac{1}{2}\eta_0^4\xi^2\eta^2 - \frac{2U_\infty}{\ln Pr}\psi(\xi,\eta)\left(1+\left(\frac{A}{\ln Pr}\right)+\left(\frac{A}{\ln Pr}\right)^2+\dots\right) \\ &\quad + \frac{1}{Pr\ln Pr}\Psi_1(\xi,\eta) + \frac{1}{Pr^2\ln Pr}\Psi_2(\xi,\eta)+\dots \\ &= \tfrac{1}{2}\eta_0^4\xi^2 - \frac{2U_\infty}{\ln\dfrac{2Pr}{\eta_0^2(1+U_\infty)}+1-\gamma_0}\psi(\xi,\eta) \\ &\quad + \frac{1}{Pr\ln Pr}\Psi_1(\xi,\eta) + \frac{1}{Pr^2\ln Pr}\Psi_2(\xi,\eta)+\dots.\end{aligned} \tag{5.9}$$

Similarly, we derive the outer expansion solution:

$$\begin{aligned}\hat{\Psi}(\xi,\eta) &= 2Pr^2\eta_0^4(1+U_\infty)\,\xi_*^2\,\eta_*^2 \\ &\quad + \frac{2U_\infty}{1+U_\infty}\frac{Pr^2}{\ln Pr}\phi(\xi_*,\eta_*)\left(1+\left(\frac{A}{\ln Pr}\right)+\left(\frac{A}{\ln Pr}\right)^2+\dots\right) \\ &\quad + \frac{Pr}{\ln Pr}\hat{\Psi}_2(\xi_*,\eta_*) + \frac{1}{\ln Pr}\hat{\Psi}_3(\xi_*,\eta_*)+\dots \\ &= 2Pr^2\eta_0^4(1+U_\infty)\,\xi_*^2\,\eta_*^2 + \frac{2U_\infty}{1+U_\infty}\frac{Pr^2}{\ln\left[\dfrac{2Pr}{\eta_0^2(1+U_\infty)}\right]+1-\gamma_0}\phi(\xi_*,\eta_*) \\ &\quad + \frac{Pr}{\ln Pr}\tfrac{1}{2}U_\infty\,\eta_0^2(\sigma_*+\tau_*) + \frac{3}{\ln Pr}\tfrac{1}{4}U_\infty\,\eta_0^2+\dots.\end{aligned} \tag{5.10}$$

6. Asymptotic solution for temperature field and interface shape

We now derive the asymptotic expansion solutions for the temperature field $T(\xi, \eta)$ and the interface shape $\eta_s(\xi)$. According to the inner expansion solution (5.9), we write the following expansions:

$$T(\xi,\eta) = T_*(\eta) + \frac{2U_\infty}{\ln\left[\dfrac{2Pr}{\eta_0^2(1+U_\infty)}\right]+1-\gamma_0} T_0(\xi,\eta) + \frac{1}{Pr\ln Pr} T_1(\xi,\eta) + \frac{1}{Pr^2\ln Pr} T_2(\xi,\eta) + \dots, \tag{6.1}$$

$$\eta_s(\xi) = 1 + \frac{2U_\infty}{\ln\left[\dfrac{2Pr}{\eta_0^2(1+U_\infty)}\right]+1-\gamma_0} h_0(\xi) + \frac{1}{\Pi\rho\ln Pr} h_1(\xi) + \frac{1}{Pr_2\ln Pr} h_2(\xi) + \dots. \tag{6.2}$$

In the following, we shall derive the zeroth-order expansion solutions $(\delta_0(Pr)\, T_0(\xi, \eta))$, where we set

$$\delta_0(Pr) = \frac{2U_\infty}{\ln\left[\dfrac{2Pr}{\eta_0^2(1+U_\infty)}\right]+1-\gamma_0}. \tag{6.3}$$

The zeroth-order expansion solution $\delta_0(Pr)\, T_0(\xi, \eta)$ is subject to the following equation:

$$\nabla^2 T_0 = \frac{1}{\eta_0^2\xi\eta}\left\{\frac{\partial\Psi_*}{\partial\eta}\frac{\partial T_0}{\partial\xi} - \frac{\partial\Psi_*}{\partial\xi}\frac{\partial T_0}{\partial\eta} - \frac{\mathrm{d}T_*}{\mathrm{d}\eta}\frac{\partial\Psi_0}{\partial\xi}\right\}, \tag{6.4}$$

with the boundary conditions:

the far-field condition

$$\text{as } \eta\to\infty \quad T_0 \to 0; \tag{6.5}$$

the interface condition at $\eta = 1$:

$$T_0(\xi, 1) = \eta_0^2 h_0 \tag{6.6}$$

$$\frac{\partial T_0}{\partial\eta} = -\eta_0^2(2+\eta_0^2)\, h_0 - \eta_0^2\,\xi h_0'(\xi). \tag{6.7}$$

It has already been derived that

$$\left.\begin{aligned} \frac{\mathrm{d}T_*}{\mathrm{d}\eta} = -\frac{\eta_0^2}{\eta} e^{\frac{1}{2}\eta_0^2 - \tau}, \quad \frac{\partial\Psi_0}{\partial\xi} = \tfrac{1}{2}\eta_0^4\,\xi f(\eta), \\ f(\eta) = 2\eta^2\ln\eta - \eta^2 + 1. \end{aligned}\right\} \tag{6.8}$$

Therefore, we may assume that the solution T_0 only depends on the variable η, namely $T_0 = T_0(\eta)$: while h_0 is just a constant. Consequently, we derive the governing equation

$$T_0''(\eta) + \left(\frac{1}{\eta} + \eta_0^2\eta\right) T_0'(\eta) = \frac{\eta_0^4}{2\eta^2} e^{\frac{1}{2}\eta_0^2(1-\eta^2)} f(\eta) \tag{6.9}$$

with the boundary conditions:

$$\text{as} \quad \eta\to\infty \quad T_0\to 0; \tag{6.10}$$

at $\eta = 1$:

$$T_0(1) = \eta_0^2 h_0, \tag{6.11}$$

$$T_0'(1) = -\eta_0^2(2+\eta_0^2)\, h_0. \tag{6.12}$$

This free boundary problem can be easily solved. The solution is as follows:

$$T_0(\eta) = \eta_0^4 e^{\frac{1}{2}\eta_0^2}[Q(\eta) + B_0 R(\eta)], \tag{6.13}$$

where

$$\left.\begin{aligned} Q(\eta) &= \int_\eta^\infty [\tfrac{1}{2}t(1-\ln t) - (\ln t/t] e^{-\frac{1}{2}\eta_0^2 t^2}\, dt \\ &= \frac{e^{-\frac{1}{2}\eta_0^2\eta^2}}{2\eta_0^2}[1 + \ln \tfrac{1}{2}\eta_0^2 \eta] - \tfrac{1}{4}E_1(\tfrac{1}{2}\eta_0^2\eta^2)(\tfrac{1}{2}\eta_0^2 - \ln\tfrac{1}{2}\eta_0^2) \\ &\quad -\frac{1}{2}\int_{\frac{1}{2}\eta_0^2\eta^2}^\infty \frac{e^{-u}\ln u}{u}\, du, \\ R(\eta) &= \int_\eta^\infty \frac{e^{-\frac{1}{2}\eta_0^2 t^2}}{t}\, dt = \tfrac{1}{2}E_1(\tfrac{1}{2}\eta_0^2\eta^2), \end{aligned}\right\} \tag{6.14}$$

and

$$h_0 = H_0(\eta_0^2), \quad H_0 = \frac{\eta_0^2}{2+\eta_0^2}(\tfrac{1}{2} + B_0), \quad B_0 = \frac{\dfrac{\frac{1}{2}e^{-\frac{1}{2}\eta_0^2}}{2+\eta_0^2} - Q(1)}{R(1) - \dfrac{e^{-\frac{1}{2}\eta_0^2}}{2+\eta_0^2}}. \tag{6.15}$$

Finally, we obtain the asymptotic expansion solutions for the temperature field and the interface shape as follows:

$$T(\xi,\eta) = T_*(\eta) + \delta_0(Pr)\, T_0(\eta) + \frac{1}{Pr\ln Pr} T_1(\xi,\eta) + \frac{1}{Pr^2 \ln Pr} T_2(\xi,\eta) + \dots, \tag{6.16}$$

$$\eta_s(\xi) = 1 + \delta_0(Pr)\, h_0 + \frac{1}{Pr\ln Pr} h_1(\xi) + \frac{1}{Pr^2\ln Pr} h_2(\xi) + \dots. \tag{6.17}$$

It should be pointed out that the solution (6.16) and (6.17) is a uniformly valid asymptotic expansion solution in the whole region. It satisfies both the interface condition and the far-field condition. Therefore we do not need to look for the outer solution for the temperature field in the outer region. In order to make the outer expansion for the temperature field, it can be shown that this outer expansion will be T_∞ plus some exponentially small term as $Pr \to \infty$.

From the above solutions (6.16) and (6.17), one sees that the steady-state solution of dendrite growth with external flow can never be a similarity solution. This conclusion has also been drawn by Ananth & Gill through a different approach. However, if one neglects all terms of $O(1/Pr\ln Pr)$, the shape of the dendrite will be a paraboloid of revolution, while the temperature field will be described by a similarity type of solution, just like in the case of dendrite growth with no external flow. Thus, when the surface tension equals zero and the Prandtl number is large, we can consider the steady-state solution of dendritic growth with external flow as nearly a similarity solution, which can be approximated by the similarity solution $\{T_B(\eta), \eta_B\}$:

$$T_B(\eta) = T_*(\eta) + \delta_0(Pr)\, T_0(\eta), \tag{6.18}$$

$$\eta_B(\eta) = 1 + \delta_0(Pr)\, h_0. \tag{6.19}$$

From here, we can deduce that for a given undercooling T_∞, the steady solution of dendrite growth with the inclusion of external flow is different from the classical Ivantsov solution in two aspects:

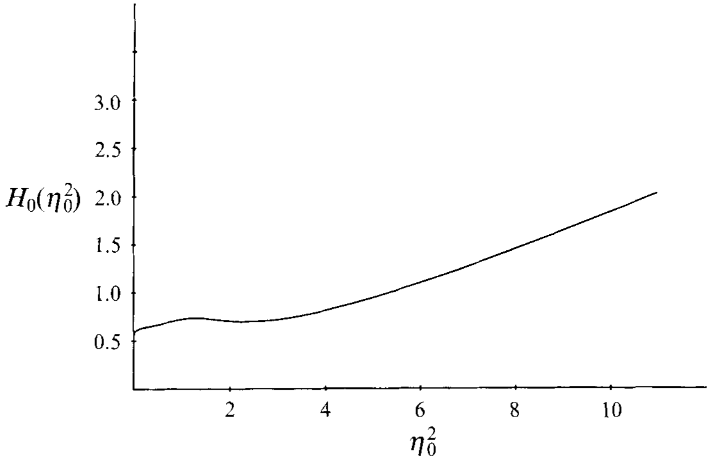

FIGURE 3. The variation of the function $H_0(\eta_0^2)$ with the parameter η_0^2.

(i) the interface shape is changed from the paraboloid $\eta_s = 1$ to the paraboloid

$$\eta_s = \eta_B = 1+\delta_0 H_0(\eta_0^2), \tag{6.20}$$

where we have defined that

$$\delta_0 = \frac{2U_\infty}{\ln\left[\dfrac{2Pr}{\eta_0^2(1+U_\infty)}\right]+1-\gamma_0}; \tag{6.21}$$

this change will affect the tip radius and the tip velocity of the dendrite;

(ii) the temperature gradient at the interfae is changed from $T'_*(1) = -\eta_0^2$ to

$$\begin{aligned} T'_B(\eta_B) &\approx T'_*(1)+T''_*(1)(\eta_B-1)+\delta_0(Pr)\,T'_0(1) \\ &= -\eta_0^2[1+\delta_0 H_0(\eta_0^2)]; \end{aligned} \tag{6.22}$$

this change will affect the instability mechanism of the dendrite interface.

The function $H_0(\eta_0^2)$ is plotted in figure 3 versus η_0^2. It is shown that as

$$\eta_0^2 \rightarrow 0, \; H_0(\eta_0^2) \rightarrow \tfrac{1}{2};$$

while as $\eta_0^2 \rightarrow \infty$, $H_0(\eta_0^2) \sim -\frac{1}{4}\eta_0^2$. It is seen that as long as $\eta_0^2 < 3$, one always has $0.5 \leqslant |H_0| < 1$. The parameter δ_0 is a function of the parameters U_∞, η_0^2, and Pr. In figure 4, we show the variation of δ_0 with Pr for two cases: (*a*) $U_\infty = 0.5$, $\eta_0^2 = 0.1$; (*b*) $U_\infty = 0.1$, $\eta_0^2 = 0.2$.

Moreover, from (6.18) and (6.19) one see that the nearly similarity solution is, actually, a modified Ivantsov solution with the correction terms proportional to the parameter δ_0, caused by the external flow. When the parameter δ_0 is small, the similarity solution (6.18) and (6.19) will be close to the Ivantsov solution. This will be the case, as the flow parameter U_∞ is less than unity. However, when the flow parameter U_∞ is large, the nearly similarity solution (6.18) and (6.19) will be far away from the Ivantsov solution.

For a given T_∞, the growth Péclet number Pe defined as the ratio of the tip radius of the dendrite, l_b and the thermal length, l_T can be calculated as

$$Pe = \eta_0^2\eta_B^2 = \eta_0^2[1+\delta_0 H_0(\eta_0^2)]^2. \tag{6.23}$$

For $Pr = 13.5$ and $U_\infty = 0, 0.4, 0.8, 1.2, 2.0, 5.0$, we have calculated Pe versus $|T_\infty|$. The results are shown in figure 5. It is seen that for a fixed T_∞, the growth Péclet number

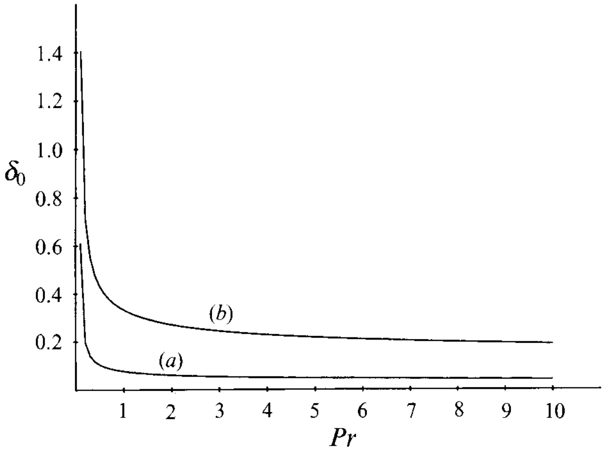

FIGURE 4. The variation of the parameter δ_0 with Pr for (a) $U_\infty = 0.1$, $\eta_0^2 = 0.2$; (b) $U_\infty = 0.5$, $\eta_0^2 = 0.1$.

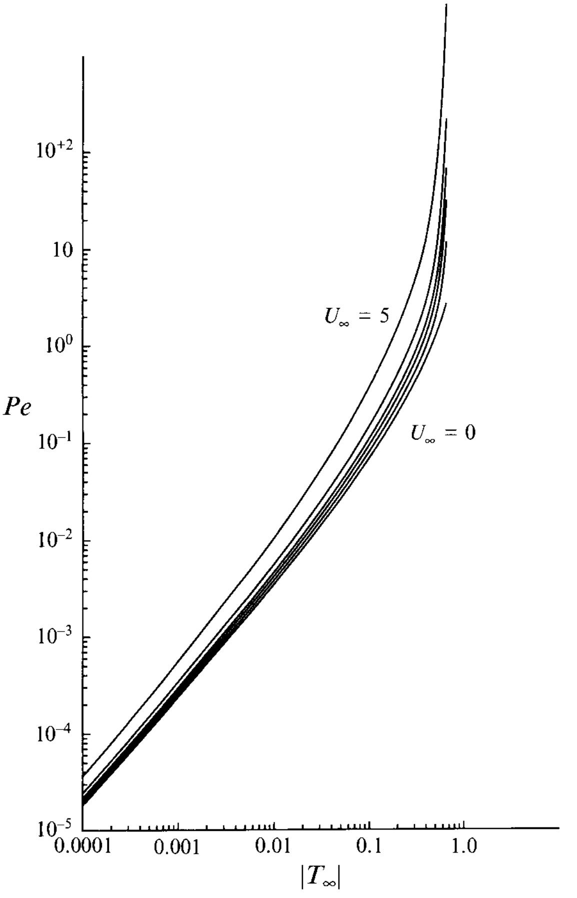

FIGURE 5. The variation of the growth parameter Pe with T_∞ for $Pr = 13.5$ and various flow parameters $U_\infty = 0, 0.4, 0.8, 1.2, 2.0, 5.0$.

Pe increases with U_∞; while for a fixed U_∞, the Péclet number Pe monotonically increases with T_∞.

The steady dendrite growth solution derived in the present paper, just like the Ivantsov solution, does not determine the tip velocity or explain the formation of micro-structure, since the surface tension in the system is neglected. The selection of the tip velocity and the formation of the pattern in dendritic growth with external flow is a subject of great significance both theoretically and practically. The solutions to these problems can be derived by using the same approach as we have developed in previous studies (see Xu 1990*a–c*, 1991, 1993). The results of our investigation will be published in a future paper.

7. Summary

In the present paper, we study steady dendrite growth from a melt in an external flow with zero surface tension at the interface. By use of a matched asymptotic expansion method, we obtain a uniformly valid asymptotic solution for the problem, as $Pr \to \infty$. The problem has been studied by a number authors in terms of Oseen, Stokes, or potential flow approximations (see Dash & Gill 1984; Ben Amar *et al.* 1988; Bouisson & Pelce 1989; Saville & Beaghton 1988; McFadden & Coriell 1986; Ananth & Gill 1989, 1991), but their solutions are not of asymptotic type. Hence, as $Pr \to \infty$ the analytical form of their solutions will not match to our solution.

The problem involves the interaction of the flow field and the temperature field. As $Pr \to \infty$, the flow field is decoupled from the temperature field. The asymptotic solutions of the flow field can be solved first. Then the asymptotic solutions of the temperature field and the interface shape function are solved based on the solutions for the flow field. In solving the flow field, however, a singularity appears at the far field. Thus, the matched asymptotic expansion method is needed. The whole physical space is divided into an inner region near the interface of the dendrite and an outer region in the far field. In different regions, the solutions to the flow field have different asymptotic expansion forms. The inner expansion solutions in the inner region can satisfy all the boundary conditions at the interface, but cannot satisfy the flow condition at the far field, whereas the outer expansion solutions in the outer region can satisfy all boundary conditions at the far field, but cannot satisfy the boundary conditions at the interface. Therefore, it is necessary to match both solutions with each other in the intermediate region in deriving the uniformly valid asymptotic expansion solution to the problem. Our matched asymptotic solution is derived in the limit $Pr \to \infty$, so it can be applied to practical cases with any undercooling parameter T_∞ and a large flow parameter. Although in figure 5 we only show numerical results in the range of U_∞ up to 5, it does not imply that the applicability of our solution has this limitation.

The conclusions drawn in the present paper are summarized as follows:

(i) With the inclusion of external flow, the system does not permit a similarity solution even for the special case of zero surface tension. However, when $Pr \to \infty$, the steady-state solution of dendrite growth is nearly a similarity solution. It can be approximated by the similarity solution $T_B(\eta)\,\eta_B$ with an error or $O(1/Pr \ln Pr)$, as described by (6.18) and (6.19).

(ii) If we neglect the higher-order small terms of $O(1/Pr \ln Pr)$ due to the effect of convection, the interfae shape of the dendrite is changed to a paraboloid $\eta = \eta_B > 1$, from the Ivantsov paraboloid $\eta = 1$, corresponding to the case $U_\infty = 0$, while the gradient of temperature at the interface is also changed as shown by (6.22);

(iii) The correction terms in the nearly similarity solution caused by the external flow are proportional to the parameter δ_0, which is a function of Pr, U_∞, T_∞. As $Pr \rightarrow \infty$ or $U_\infty \rightarrow 0$, $\delta_0 \rightarrow 0$. Therefore, when the flow parameter U_∞ is small, the correction terms are small. However, when the flow parameter is large ($U_\infty \gg 1$), the correction terms in the nearly similarity solution may be large. Hence, our solution may be quite different from the Ivantsov solution.

(iv) For any given T_∞, the growth Péclet number Pe increases with the flow parameter U_∞. For a fixed U_∞, the Péclet number Pe increases monotonically with T_∞.

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REFERENCES

ABRAMOVITZ, M. & STEGUN, I. A. (ed.) 1964 *Handbook of Mathematical Functions*. National Bureau of Standards.

ANANTH, R. & GILL, W. N. 1989 *J. Fluid Mech.* **208**, 575.

ANANTH, R. & GILL, W. N. 1991 *J. Cryst. Growth* **108**, 173.

BEN AMAR, M., BOUISSOU, PH. & PELCE, P. 1988 *J. Cryst. Growth* **92**, 97.

BOUISSOU, PH. & PELCE, P. 1989 *Phys. Rev.* A **40**, 6673.

CANRIGHT, D. & DAVIS, S. H. 1990 *Applied Mathematics Tech. Rep.* 8982. Northwestern University.

DASH, S. K. & GILL, W. G. 1984 *Intl J. Heat Mass Transfer* **27**, 1345.

HUANG, S. C. & GLICKSMAN, M. E. 1981 *Acta Metall.* **29**, 701.

IVANTSOV, G. P. 1947 *Dokl. Nauk. SSSR* **58**, 567.

KEVORKIAN, J. & COLE, J. D. 1981 *Perturbation Methods in Applied Mathematics*. Applied Mathematical Sciences, Vol. 34. Springer.

MCFADDEN, G. B. & CORIELL, S. R. 1986 *J. Cryst. Growth* **74**, 507.

SAVILLE, D. A. & BEAGHTON, P. J. 1988 *Phys. Rev.* A **37**, 3423.

XU, J. J. 1990*a* *Can. J. Phys.* **68**, 58.

XU, J. J. 1990*b* *Physica Status Solidi* (*b*), **157**, 577.

XU, J. J. 1990*c* *J. Cryst. Growth* **100**, 481.

XU, J. J. 1991 *Phys. Rev.* A.

XU, J. J. 1993 Effect of convection motion in melt induced by density change on dendritic growth. *Can. J. phys.* (to appear).